ISRAEL JOURNAL OF MATHEMATICS 82 (1993), 341–362

THE LAST FLAG-TRANSITIVE P-GEOMETRY

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Dedicated to Prof. J. Thompson

ABSTRACT

The universal 2-cover of the P-geometry related to the Baby Monster sporadic simple group BM is shown to admit a non-split extension $3^{4371} \cdot BM$ as a flag-transitive automorphism group. This new geometry completes the list of flag-transitive P-geometries.

1. Introduction

In this paper we construct a new flag-transitive P-geometry and prove its simple connectedness. This answers the last question being open in the classification project of the flag-transitive P-geometries (cf. [8, 9, 14, 15]). The constructed geometry closes the list of the flag-transitive P-geometries which is now known to consist of eight examples: $\mathcal{G}(M_{22})$, $\mathcal{G}(3 \cdot M_{22})$, $\mathcal{G}(M_{23})$, $\mathcal{G}(Co_2)$, $\mathcal{G}(3^{23} \cdot Co_2)$, $\mathcal{G}(J_4)$, $\mathcal{G}(BM)$ and $\mathcal{G}(3^{4371} \cdot BM)$.

The basic definitions can be found in [3]. P-geometries are the geometries belonging to a connected string diagram whose non-empty edges are all projective planes over $GF(2)$, except the rightmost edge which stands for the geometry of

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Received September 3, 1992 and in revised form October 19, 1992

edges and vertices of the Petersen graph, with the natural incidence relation. In what follows all the considered geometries are flag-transitive, by assumption or by construction. So, sometimes we simply write "geometries" meaning them all flag-transitive. If $\mathcal{G}$ is a geometry over a type set Δ, and $i \in \Delta$, then $\mathcal{G}^i$ denotes the set of elements of type i in the geometry. For an element $x \in \mathcal{G}$, its residue is denoted by $\text{res}(x)$. Thus, $\text{res}(x)^i$ is the set of elements of type i incident to x.

The new P-geometry will be constructed as a 2-cover (actually, the universal 2-cover) of the rank 5 P-geometry $\mathcal{G}(BM)$ related to Fischer's Baby Monster sporadic simple group BM. The latter P-geometry was first constructed in [7], and its simple connectedness was proved in [8]. In the present paper we compute the universal 2-cover of $\mathcal{G}(BM)$. In general, the universal 2-cover of a geometry is not necessarily itself a geometry. Nevertheless, by means of some standard arguments relying on the shape of the diagram and the structure of residues (see [1]), one can prove that the 2-cover of a flag-transitive P-geometry is always a geometry.

MAIN THEOREM: *Let $\mathcal{G} = \mathcal{G}(BM)$ and $\tilde{\mathcal{G}}$ be its universal 2-cover. Then the group K of the deck transformations of $\tilde{\mathcal{G}}$ with respect to the mapping $\tilde{\mathcal{G}} \longrightarrow \mathcal{G}$ is elementary abelian of order 3^{4371}. In particular, $\tilde{\mathcal{G}}$ admits a flag-transitive automorphism group which is a non-split extension $3^{4371} \cdot BM$. The factor group BM of the latter group acts irreducibly on K.*

Remark: In fact, BM is the only flag-transitive automorphism group of $\mathcal{G}(BM)$, and hence $3^{4371} \cdot BM$ is the only flag-transitive automorphism group of $\tilde{\mathcal{G}}$. It means that $\mathcal{G}$ is the only flag-transitive factor of $\tilde{\mathcal{G}}$, so that no further flag-transitive P-geometry can be obtained as a factor of $\tilde{\mathcal{G}}$.

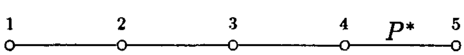
In its basic features the approach of this paper resembles that of [15]. However, in the present paper the proof goes much faster. In the construction part we use computations with the character table of BM, which are now feasible, and even comfortable due to new Version 3.1 of GAP. On the other hand, when proving that the automorphism group of $\tilde{\mathcal{G}}$ is an abelian extension of BM, and bounding the rank of the kernel, we use simple geometric arguments in the spirit of presheaves [13], but in the characteristic which is not natural for the geometry. These considerations are based on the module structure of the kernels arising in various subgeometries of $\mathcal{G}(BM)$. The relevant information was established in [10, 15].

Special words must be said about the relation between this paper and [17]. In the construction part, there is a key Proposition 3.1. By this proposition, a certain 2-subgroup Q_5 of BM acts fixed-points-freely on an explicitely defined $GF(3)$-module of BM. The natural way to check this basic property is via computations with the character table. However, these computations require a precise information about the conjugacy classes of BM within Q_5. This information is not available to us even now, so one of the authors applied to R.A.Wilson, who had just constructed astonishing matrices giving a representation of BM of dimension 4370 over $GF(2)$. The request was to check the property of Q_5 computationally, via these matrices. Such a check has been performed during last months. When it was more or less completed, a trick (see Corollary 3.11) was found, letting substitute Q_5 by another subgroup, more convenient for the computations with characters. Here we give this alternative proof of the existence of a non-trivial 2-cover of $\mathcal{G}(BM)$.

2. Preliminaries-1

The necessary information about the Baby Monster group BM and its geometry will be given in two portions. In this short section we collect very few facts about the parabolic subgroups of $G = BM$ corresponding to its action on $\mathcal{G} = \mathcal{G}(BM)$, which are needed for our construction of a nontrivial 2-cover of $\mathcal{G}$. All the subgroups arising in this way are well-known. The best reference is [4]. More details of residues and subgeometries of $\mathcal{G}$ will be given in Section 4.

The group G contains an elementary abelian subgroup $E \cong 2^5$, such that $N_G(E) \cong [2^{30}].L_5(2)$ and $N_G(E)$ induces on E its full automorphism group $L_5(2)$. It was shown in [9] that the set of subgroups of G, conjugate to nontrivial subgroups of E, constitutes a geometry with the following diagram



where the type of an element is given by its rank, and the rightmost edge stands for the dual of the Petersen graph geometry. It means that $\mathcal{G}$ is a rank 5 P-geometry (see [9]). The Borel subgroup B in G is a 2-group of order 2^{40} (index 2 in the Sylow 2-subgroup). Let P_i, $i = 1, ..., 5$, be the minimal parabolic subgroups. Then P_1 to P_4 have the shape $[2^{39}].S_3$, while P_5 is a 2-group of order 2^{41}. Now let $P_{ij} = \langle P_i, P_j \rangle, 1 \leq i < j \leq 5$, be the rank 2 parabolic subgroups. The structure of P_{ij} is basically determined by the diagram, since P_{ij} acts on the corresponding rank 2 residue of type $\{i,j\}$. The only case to be specified is that $P_{4,5}/O_2(P_{4,5}) \cong S_5$, since S_5 (not A_5) is the flag-transitive group induced on the corresponding residue, isomorphic to the (dual) Petersen geometry.

Finally, let H_i, $i = 1, \dots, 5$, denote the i-th maximal parabolic subgroup, i.e., the stabilizer of the element x_i of type i from the chosen maximal flag of $\mathcal{G}$. Without loss of generality we may assume that x_i is a subgroup of order 2^i in E (so that $E = x_5$). Since x_i's form a flag we have that $x_i < x_j$ whenever $i < j$. In these terms P_i is the stabilizer of all x_j, $j \neq i$.

The structures of H_i's are given by:

$H_1 \cong 2^{1+22} \cdot Co_2$,
$H_2 \cong [2^{32}].(S_3 \times \mathrm{Aut}\, M_{22})$,
$H_3 \cong [2^{35}].(L_3(2) \times S_5)$,
$H_4 \cong [2^{34}].(L_4(2) \times 2)$,
$H_5 \cong [2^{30}].L_5(2)$.

Let $Q_i = O_2(H_i)$, $i = 1, \dots, 5$. Unless $i = 4$, Q_i is the kernel of the action of H_i on $\mathrm{res}(x_i)$. The group Q_4 is twice larger than the corresponding kernel, since it also involves the action on the two elements of type 5 incident to x_4. Particularly, it means that $Q_5 \leq Q_4$.

The following result is a consequence of the information in [4].

Lemma 2.1: $N_G(Q_i) = H_i$.

3. A non-trivial 2-cover

Let $\mathcal{P} = \{P_{ij}\}_{1 \leq i < j \leq 5}$ be the amalgam of the rank 2 parabolic subgroups related to the action of $G = BM$ on $\mathcal{G} = \mathcal{G}(BM)$. Whenever $\mathcal{P}$ is embedded in a group, it defines for that group a transitive chamber system with the diagram as above. By a standard argument this chamber system must be a geometry. Moreover, this new P-geometry has the same universal 2-cover, as $\mathcal{G}$ does, and the universal 2-cover itself can be obtained in this way. In this section we embed $\mathcal{P}$ in a certain group and prove that the subgroup, generated by $\mathcal{P}$, *is not isomorphic* to G. Clearly, it implies that $\mathcal{G}$ is not 2-simply connected.

It is well-known (see [4]) that G has a subgroup L of the shape $2 \cdot {}^2E_6(2).2$. This subgroup is the centralizer of a 2A-involution from G, according to the notation of [4]. Let U_0 be the *nontrivial* 1-dimensional module for L over $GF(3)$.

Then $L' \cong 2 \cdot {}^2E_6(2)$ acts trivially on U_0. Let U be the G-module induced from U_0. The 3-part of the Schur multiplier of ${}^2E_6(2)$ has order 3. Moreover, every outer automorphism of order 2 of ${}^2E_6(2)$ acts on that 3-part nontrivially. This can be seen from the 27-dimensional irreducible representation of $3 \cdot {}^2E_6(2)$ over $GF(4)$, because the outer involution can be represented in this case by the field automorphism. In particular, the second cohomology group $H^2_L(U_0)$ is nontrivial of order 3. Now by Eckmann-Shapiro lemma (see [2], Shapiro lemma), also $H^2_G(U)$ has order 3. It means that there is a (unique) non-split extension $\hat{G} = U \cdot G$. It is the group into which we are going to embed $\mathcal{P}$. Since the extension is non-split, such an embedding can not generate a subgroup isomorphic to $G = BM$. Notice that the group $\hat{G}$ is much larger than the group $3^{4371} \cdot BM$ from Main Theorem. In particular, $\mathcal{P}$ must generate a proper subgroup of $\hat{G}$.

Our construction is based on the following key fact.

PROPOSITION 3.1: *The group Q_5 fixes no nontrivial vector in U, i.e., $C_U(Q_5) = 0$.*

Before proving this proposition we show how it yields an embedding of $\mathcal{P}$ into $\hat{G}$. First of all, since $Q_5 \leq Q_4$ we have the following

COROLLARY 3.2: $C_U(Q_4) = 0$.

Let $\tilde{B}$ be a 2-subgroup in $\hat{G}$ which is a complement to U in the full preimage of B with respect to the natural homomorphism of $\hat{G}$ onto G. Then, clearly, $\tilde{B}$ maps onto B isomorphically. Let $\tilde{Q}_i$, $i = 1, \dots, 5$, denote the preimage of Q_i in $\tilde{B}$. Let $\tilde{H}_5 = N_{\hat{G}}(\tilde{Q}_5)$ and $\tilde{H}_4 = N_{\hat{G}}(\tilde{Q}_4)$. Since $C_U(Q_5) = C_U(Q_4) = 0$, the Frattini argument implies that $\tilde{H}_4$ and $\tilde{H}_5$ are isomorphic to their images H_4 and H_5, respectively. The latter two subgroups contain all minimal parabolic subgroups P_i. Clearly, $\tilde{H}_4 \cap \tilde{H}_5$ maps onto $H_4 \cap H_5$, so that we can define $\tilde{P}_i$ as the preimage of P_i in a suitable subgroup $\tilde{H}_i, i = 4, 5$. Moreover, every rank two parabolic subgroup P_{ij}, except $P_{4,5}$, is contained in at least one of subgroups H_4 and H_5. Hence we can formulate

LEMMA 3.3: *Unless $\{i,j\} = \{4,5\}$, $\tilde{P}_{ij} = \langle \tilde{P}_i, \tilde{P}_j \rangle$ is isomorphic to P_{ij}.*

We are going to prove that the same statement is valid in the case $\tilde{P}_{4,5}$ as well. Let $\mathcal{Q}$ be the amalgam $\{2 \times S_3, D_8\}$ arising in the action of S_5 on the Petersen geometry. The following fact will be useful.

LEMMA 3.4: *Whenever embedded into a group $F \cong 3^s.S_5$ with trivial action of $A_5 \triangleleft F/O_3(F)$ on $O_3(F)$, the amalgam $\mathcal{Q}$ generates a subgroup S_5.*

Proof: Since the 3-part of the Schur multiplier of A_5 is trivial, F has a normal subgroup A_5. Consider the factor group of F over this A_5. It is easy to see that the whole amalgam $\mathcal{Q}$ maps onto a group of order 2, so the claim follows. ∎

LEMMA3.5: *$\tilde{P}_{4,5}$ is isomorphic to $P_{4,5}$.*

Proof: Let $Q = O_2(P_{4,5})$ and $R = O_2(P_4)$. Let $\tilde{Q}$ and $\tilde{R}$ be the corresponding preimages in $\tilde{B}$. Note that the image of $\{P_4, P_5\}$ in $P_{4,5}/Q$ is just the amalgam $\mathcal{Q}$. In particular, $|R : Q| = 2$. Since R contains Q_5, $C_U(R) = 0$. It follows that an element from $R \setminus Q$ inverts all elements of $C_U(Q)$. It means that $P_{4,5}$ has only 1-dimensional factors within $C_U(Q)$, i.e., the subgroup A_5 of $P_{4,5}/Q \cong S_5$ acts trivially on $C_U(Q)$. Now, the group $\tilde{P}_{4,5}/\tilde{Q}$ is contained in $N_{\hat{G}}(\tilde{Q})/\tilde{Q} \cong 3^s.S_5$, where 3^s stands for the image of $C_U(Q)$. Since the image of $\tilde{P}_{4,5}$ in $\tilde{P}_{4,5}/\tilde{Q}$ is generated by the same amalgam $\mathcal{Q}$, it must be isomorphic to S_5 in view of Lemma 3.4. ∎

Lemmas 3.3 and 3.5 provide an embedding of the amalgam $\mathcal{P} = \{P_{ij}\}_{1 \le i < j \le 5} \cong \{\tilde{P}_{ij}\}_{1 \le i < j \le 5}$ into $\hat{G}$. As $\hat{G}$ is a non-split extension, the closure $\langle \{\tilde{P}_{ij}\}_{1 \le i < j \le 5} \rangle$ is not isomorphic to $G = BM$. We have shown that the following proposition, constituting the main result of the section, follows from Proposition 3.1.

PROPOSITION 3.6: *The geometry $\mathcal{G}(BM)$ is not 2-simply connected.*

The proof of Proposition 3.1 will be given in a sequence of lemmas. We start with some general comments on calculation of centralizers in a module induced from a non-trivial 1-dimensional module. Namely, we specialize for our particular situation the Mackey decomposition (see [5], Mackey theorem). Lemmas 3.7 and 3.8 are apparently well-known; yet we prove them for the sake of completeness.

Let F be a group and X, Y be subgroups of F. Let W_0 be a non-trivial 1-dimensional module of X over some field, and W the F-module induced from W_0. We are interested in the dimension of $C_W(Y)$. Let X_0 be the kernel of X acting on W_0. Since W is the induced module, it is a direct sum $\oplus_{i \in \mathcal{T}} W_i$ of 1-dimensional subspaces indexed by the cosets from $\mathcal{T} = F/X$. The group F naturally permutes the subspaces W_i, and if we put i_0 to be the coset $1 \cdot X$, then W_{i_0} is isomorphic to W_0 as an X-module. For every $i = gX$ let $X_i = X^g$ be the stabilizer, and $X_{0,i} = X_0^g$ be the centralizer of W_i. Choose an orbit T of Y on $\mathcal{T}$ and let $W_T = \bigoplus_{i \in T} W_i$. Then dim $C_{W_T}(Y) = 1$, or 0, depending on whether, for $i \in T$, $Y \cap X_i$ is contained in $X_{0,i}$, or is not. We will say that T is *non-twisted* and *twisted* in the respective cases. Clearly, $C_W(Y)$ is the direct sum of 1-dimensional subspaces $C_{W_T}(Y)$ over all non-twisted Y-orbits T.

We have proved the following

LEMMA 3.7: *In the above notation,* dim $C_W(Y)$ *is equal to the number of non-twisted orbits of* Y *on* $\mathcal{T}$.

Let us also give a character theoretical interpretation of this lemma. Notice that the above definition does not depend on the characteristic of the ground field, but only on the pair (X, X_0). Let $\tilde{W}_0$ be the non-trivial 1-dimensional X-module over $\mathbf{C}$ with the same kernel X_0, and $\tilde{W}$ be the F-module induced from $\tilde{W}_0$. Then Lemma 3.7 gives us that dim $C_W(Y) =$ dim $C_{\tilde{W}}(Y)$. On the other hand, the latter dimension is simply the number of trivial subconstituents of the character of $\tilde{W}$, restricted to Y. We formulate this as

LEMMA 3.8: *In the above notation, let* χ_0 *be the character of* $\tilde{W}_0$, *and* $\chi = \chi_0^F$ *the induced character. Then* dim $C_W(Y) = (\chi|_Y, 1_Y)$.

Due to Lemma 3.8 we can work with the usual character table of $G = BM$. In what follows if W is a module induced from a 1-dimensional module, then by the character of W we always mean the complex character related with W as above.

Now let us turn to the proof of Proposition 3.1. Let χ_0 be the only 1-dimensional non-trivial character of L and χ the corresponding induced character of G. The character χ was determined in [6].

LEMMA 3.9: *The character* χ, *associated as above with the action of* G *on* U, *is the sum of the irreducible characters* χ_2, χ_7 *and* χ_{17}, *according to the notation of* [4].

According to Lemma 3.8, since the character χ is known, the natural way to compute dim $C_U(Q_5)$ would be via computation of the mean value of χ on Q_5. Unfortunately, we do not know the intersections of the conjugacy classes of G with Q_5. So, we use a somewhat tricky way to obtain the desired result.

Let us substitute Q_5 by another subgroup, namely, by $R = O_2(H_1 \cap H_5)$. Clearly, R contains both Q_5 and Q_1. Let us show that $C_U(Q_5) \neq 0$ would imply $C_U(R) \neq 0$. We will use the following

LEMMA 3.10: *Let $F = L_5(2)$ and $E = O_2(P) \cong 2^4$, where P is the stabilizer of a point of the projective geometry associated with F. Then, whenever W is an F-module induced from a 1-dimensional module of a subgroup $T \leq F$, one has $C_W(E) \neq 0$.*

Proof: If W is a permutational module, then $C_W(E) \neq 0$, since even $C_W(F) \neq 0$. Hence we can assume that W is induced from a nontrivial 1-dimensional module of T. Let T_0 be the kernel of that 1-dimensional representation of T. Suppose to the contrary that $C_W(E) = 0$. Then, by Lemma 3.7, every orbit of E on the cosets of T is twisted, that is $E^x \cap T \not\leq T_0$ for every $x \in F$.

The list of maximal subgroups of $L_5(2)$ [4] consists of the maximal parabolic subgroups and the group F_{31}^5. Let M be a maximal subgroup of F containing T. Since F_{31}^5 is a $2'$-group, conjugates of E intersect it trivially. It follows that $M \not\cong F_{31}^5$. Hence M is a maximal parabolic subgroup, i.e. the stabilizer of a subspace V_0 of the basic 5-dimensional space V acted on by F. Recall that P is the stabilizer of a point, that is a 1-subspace $Z < V$. It implies that E is the group of all transvections with center Z.

Suppose first that V_0 is a 1-subspace, i.e., without loss of generality we may assume that $V_0 = Z$ and, hence, $E = O_2(M)$. If T acts on E irreducibly, then, clearly, each of $E \cap T$ and $E \cap T_0$ is either E, or 0. Since T/T_0 is cyclic, it implies $E \cap T = E \cap T_0$. If the action of T on E is reducible, then T stabilizes a proper subspace of V of dimension more than 1, which means that T is contained as well in a maximal parabolic subgroup of a different type.

Suppose now that V_0 is a hyperplane in V. Then there is $x \in F$, such that $Z^x \not\leq V_0$. In particular, $E^x \cap M = 1$, a contradiction.

Next, suppose V_0 is a 3-dimensional subspace. Choose x such that $Z^x \not\leq V_0$. Then $E^x \cap M$ is a group of order 2. By the above condition, T contains this group, as well as all its conjugates in M. In particular, $O_2(M) \leq T$ and in the factor group $M/O_2(M) \cong S_3 \times L_3(2)$ the image of T covers the direct factor S_3. Since a 3-element from this direct factor acts on $O_2(M)^\#$ fixed-point-freely, $O_2(M) \leq T_0$. Let us now choose x, such that $Z^x \leq V_0$. Then $E^x \leq M$ and the image of E^x in $\bar{M} = M/O_2(M)$ belongs to the subgroup $D \cong L_3(2) \leq \bar{M}$ and has rank 2. It implies that for $A = \bar{E^x}$, $X = \bar{T} \cap D$, $X_0 = \bar{T_0} \cap D$ we have the same condition as for E, T and T_0: $A^d \cap X \not\leq X_0$ for every $d \in D$. Applying the above two arguments (for V_0 a 1-space, and a hyperplane) to $D \cong L_3(2)$, we establish that D does not contain such a pair X, X_0.

Finally, let us suppose that V_0 is a 2-subspace. Let V' be a complement of V_0 in V, and let $D \cong L_3(2)$ be the group stabilizing V' and acting trivially on V_0. If we take x such that $Z^x \leq V'$, then $E^x \cap M \leq D$. Take once again $X = T \cap D$ and $X_0 = T_0 \cap D$. Then for the subgroups $A = E^x \cap D$, X and X_0 of D we have once again the condition: $A^d \cap X \not\leq X_0$ for every $d \in D$. As above, this forces a contradiction. ∎

COROLLARY 3.11: *If $C_U(Q_5) \neq 0$ then $C_U(R) \neq 0$.*

Proof: Suppose $W = C_U(Q_5)$ is nontrivial. By the Mackey decomposition (see the discussion before Lemma 3.7), W is a direct sum of 1-dimensional Q_5-modules, corresponding to the non-twisted Q_5-orbits on the cosets of L in G. In particular, as a H_5/Q_5-module, W is a direct sum of modules, induced from 1-dimensional modules of certain subgroups of H_5/Q_5. The image of $H_1 \cap H_5$ in $H_5/Q_5 \cong L_5(2)$ is (up to the obvious duality) a point stabilizer, hence the statement follows from Lemma 3.10. ∎

We will prove $C_U(R) = 0$ in two steps. First we determine $C = C_U(Q_1)$ and then determine the fixed space of R in C (notice that Q_1 is normal in R).

LEMMA 3.12: *$C = C_U(Q_1)$ has dimension 51175.*

Proof: By Lemma 3.8, $\dim C = 1/|Q_1| \cdot \sum_{x \in Q_1} \chi(x)$. As it was said in Section 2, H_1 is the well-known maximal subgroup $2^{1+22} \cdot Co_2$, which is the centralizer of a 2B-involution from $G = BM$ (notation for the conjugacy classes as in [4]). In particular, Q_1 is an extraspecial group of order 2^{23}. The 22-dimensional $GF(2)$-module, arising on top of Q_1, is a section of $\Lambda/2\Lambda$, where Λ is the Leech lattice. The orbits of Co_2 in this module are well-known and their lengths can be read, say, from Table 1 in [16]. Finally, we obtain the following lengths of the orbits of H_1 acting on $Q_1^{\#}$ by conjugation: $1, 2 \cdot 2300, 2 \cdot 46575, 2 \cdot 476928, 2 \cdot 1619200$ and $2 \cdot 2049300$. Moreover, we can specify that the 4th and 5th orbits consist of elements of order 4, while all others consist of involutions. Clearly, the orbit 1 belongs to the class 2B of G. Let x be the element of $\mathcal{G}$ fixed by H_1. There are exactly $2 \cdot 46575$ other elements in $\mathcal{G}^1$, at distance 2 from x in the incidence graph of $\mathcal{G}$. These elements form an orbit under the action of H_1 and, to each of these elements, there is associated a 2B-involution, which clearly belongs to

Q_1. Hence the orbit $2 \cdot 46575$ is contained in 2B as well. Now if x belongs to the 4th or 5th orbit, then x^2 is the central involution in H_1. Hence we can compute the lengths of the corresponding classes of G, multiplying 953856 and 3238400 by the number of 2B-involutions. In this way we obtain that the 4th and 5th orbits belong to 4A and 4B, respectively. Now a short program written in GAP checks that the above mentioned sum takes an integer value (namely, 51175—which is what we had to prove) only if the 2nd and 6th orbits belong to the classes 2A and 2D, respectively. ∎

Considered as a module for $H_1/Q_1 \cong Co_2$, the space C is a direct sum of modules induced from 1-dimensional modules of certain subgroups $K_1, \dots, K_s$ of H_1/Q_1. We are going to determine the subgroups K_i and to specify the involved 1-dimensional modules. We already know the degree 51175 of the character of H_1/Q_1 on C. Now the list of maximal subgroups of Co_2 [16] leaves very few possibilities for the modules involved. In particular the required information would follow from the value of the character of C on, say, any class of involutions of H_1/Q_1. Unfortunately, these values are not available and once more we should use a detour. Namely, we first determine the orbits of H_1 on the cosets of L, equivalently, on the conjugacy class 2A of G. Then we check that three of those orbits contain non-twisted Q_1-orbits and finally use the value from Lemma 3.12 to prove that we have encountered already all such orbits.

Recall that H_1 is the centralizer of a 2B-involution, while L is the centralizer of a 2A-involution. Using GAP, it is easy to obtain that for a fixed 2B-involution x, there are exactly

(O_1) 4600 $y \in$2A, such that $xy \in$2A;

(O_2) 3643200 $y \in$2A, such that $xy \in$2D;

(O_3) 190771200 $y \in$2A, such that $xy \in$4A;

(O_4) 3730636800 $y \in$2A, such that $xy \in$4D;

(O_5) 9646899200 $y \in$2A, such that $xy \in$6A;

and that any 2A-involution belongs to one of the above subsets.

The first set O_1 is already known to be an orbit, since these are the 2A-involutions from Q_1. The stabilizer in H_1 of a representative of O_1 has shape $[2^{22}].U_6(2).2$.

Let y belongs to O_2. Then $y \in H_1$, but $y \notin Q_1$. To which class of $H_1/Q_1 \cong Co_2$ does the image of y belong? Because of the cardinality, it must be either 2a, or 2b (we use small letters in order to distinguish classes of Co_2 from those of BM). Let

a be the number of y's in O_2 mapping to a fixed 2a-element, while b the number of y's mapping to a fixed 2b-element. Then $a \cdot 56925 + b \cdot 1024650 = 3643200$. This has two solutions: $a = 10, b = 3$ and $a = 64, b = 0$. Let x be a 2a-involution from $H_1/Q_1 \cong Co_2$. Then $X = C_{H_1/Q_1}(x) \cong 2^{1+8}.\mathrm{Sp}_6(2)$ does not stabilize a nontrivial vector from $E = Q_1/Z(Q_1) \cong 2^{22}$ (cf. Table 1 from [16]). It implies that X acts nontrivially on $E/C_E(x)$ (isomorphic to $[E, x]$ as an X-module). Since X involves $\mathrm{Sp}_6(2)$, it is easy to see that dim $E/C_E(x) \geq 6$ and hence $|Q_1 : C_{Q_1}(y)| \geq 64$, whenever y maps to a 2a-element. It means that the second variant holds, and also that O_2 is an orbit for H_1 and the stabilizer in H_1 of an element $y \in O_2$ has shape $[2^{26}].\mathrm{Sp}_6(2)$.

Consider now O_3. GAP gives the value 200 for the number of pairs (x, y) $\in$(2A,2B), such that $z = xy$ is a fixed element from 4A. Each subgroup $\langle x, y\rangle \cong D_8$ contains two such pairs, hence there are 100 subgroups. We have already seen that $z \in O_2(C_G(t))$, where $t = z^2$ $\in$2B. It follows from, say, Table 1 from [16], that the centralizer of z has shape $[2^{22}].\mathrm{Aut}\, HS$. Since HS has no subgroup of index less than 100, it is straightforward that there is exactly one orbit of triples (x, y, z) $\in$(2A,2B,4A) with $z = xy$, and the stabilizer of a particular triple has shape $[2^{22}].\mathrm{Aut}\, M_{22}$.

For a fixed z $\in$4D (in 6A) there are exactly 2 (respectively, 3) pairs (x, y) $\in$(2A, 2B) with $xy = z$, so that already $\langle z\rangle$ acts on these pairs transitively. Hence H_1 is transitive on both O_4 and O_5. It is now easy to establish the shapes of the corresponding stabilizers. They are $[2^{16}].\mathrm{Sp}_6(2)$ and $2.U_6(2).2$. We formulate this as following

LEMMA 3.13: *The group $H_1 \cong 2^{1+22} \cdot Co_2$, acting on* 2A, *has exactly* 5 *orbits and the representative stabilizers are as follows:* $[2^{22}].U_6(2).2$, $[2^{26}].\mathrm{Sp}_6(2)$, $[2^{22}].\mathrm{Aut}\, M_{22}$, $[2^{16}].\mathrm{Sp}_6(2)$ *and* $2.U_6(2).2$.

Let us now go on with determination of the module $C_U(Q_1)$. By Lemma 3.7, it is sufficient to determine all non-twisted orbits of Q_1 on 2A. By Lemma 3.12, the total number of non-twisted Q_1-orbits is 51175. Clearly, orbits of Q_1 within one orbit of H_1 are either all twisted, or all non-twisted. We claim that the non-twisted orbits are exactly those from O_1, O_3 and O_5.

Let $i \in O_1$. Then $H_1 \cap C_G(i) \cong [2^{22}].U_6(2).2$ covers the whole maximal parabolic subgroup $2^{1+20}.U_6(2).2$ in $C_G(i)/\langle i\rangle \cong {}^2E_6(2).2$. Since $Q_1 \cap C_G(i) \leq O_2(H_1 \cap C_G(i))$, it implies that O_1 is twisted with respect to H_1, but consists of non-twisted orbits with respect to Q_1. Since $(H_1 \cap C_G(i))Q_1$ has index 2300 in H_1, we obtain a component of dimension 2300 in $C_U(Q_1)$.

Now let $i \in O_5$. Let x be the central involution of H_1. Since $ix \in$6A in this case, neither i, nor x belong to $F = H_1 \cap C_G(i) \cong 2.U_6(2).2$. Since $i \notin F$, F maps into an involution centralizer in $C_G(i)/\langle i \rangle$. Checking the orders, we see that it must be the same subgroup $2^{1+20}.U_6(2).2$. Since $x \notin F$, and since the subgroup $U_6(2)$ of Co_2 fixes a unique vector in the module $Q_1/\langle x \rangle$ (cf. Table 1 [16]), F covers the subgroup $U_6(2).2$ of Co_2. In particular, it involves an outer automorphism of $U_6(2)$. We obtain the same conclusion: O_5 is twisted with respect to H_1, but consists of non-twisted orbits with respect to Q_1. Once again, FQ_1 has index 2300 in H_1. This gives another component of dimension 2300 in $C_U(Q_1)$.

Finally, let $i \in O_3$. Then $F = H_1 \cap C_G(i)$ has shape $[2^{22}].\mathrm{Aut}\, M_{22}$. In this case $ix \in$4A. Hence F does not contain i, but its image in $C_G(i)/\langle i \rangle \cong {}^2E_6(2).2$ is contained in a centralizer of involution. Checking the orders of the involution centralizers in ${}^2E_6(2).2$, we obtain once more that it must be $2^{1+20}.U_6(2).2$. Now, $U_6(2)$ does not involve $\mathrm{Aut}\, M_{22}$. Hence, this orbit, as well, is twisted with respect to H_1, but consists of non-twisted orbits with respect to Q_1. This gives the remaining $46575 = |H_1 : FQ_1|$ dimensions of $C_U(Q_1)$. We have proved the following

LEMMA 3.14: *As a module for H_1/Q_1, $C = C_U(Q_1)$ is a direct sum of induced modules of dimensions* 2300, 2300 *and* 46575, *all three induced from non-trivial 1-dimensional modules.*

LEMMA 3.15: $C_U(R) = 0$.

Proof: It is now a simple computation, since the library of GAP contains the fusions of the character tables of the subgroups $2^{1+4+6}.L_4(2)$ (arising as the image of $H_1 \cap H_5$), $U_6(2).2$ (of index 2300) and $2^{10} : \mathrm{Aut}\, M_{22}$ (of index 46575) into the character table of $H_1/Q_1 \cong Co_2$. ∎

Proposition 3.1 follows from Lemmas 3.15 and 3.11.

4. Preliminaries-2

In this section we collect further facts about $\mathcal{G} = \mathcal{G}(BM)$, namely, we describe the universal 2-covers of the residual P-geometries of $\mathcal{G}$ and relevant 1-covers of the symplectic subgeometries of $\mathcal{G}$.

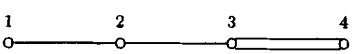
The residual P-geometries of $\mathcal{G}$ of rank 3 and 4 are isomorphic, respectively, to $\mathcal{G}(M_{22})$ and $\mathcal{G}(Co_2)$ (see [9]). The universal 2-covers of these P-geometries were determined in [15, 14].

LEMMA 4.1: (1) *The only proper cover of $\mathcal{G}(M_{22})$ is its (3-fold) universal cover $\mathcal{G}(3 \cdot M_{22})$ with the automorphism group $3 \cdot \mathrm{Aut}\, M_{22}$.*

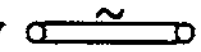
(2) *The geometry $\mathcal{G}(Co_2)$ is simply connected. It has exactly one proper flag-transitive 2-cover, which is its universal 2-cover $\mathcal{G}(3^{23} \cdot Co_2)$ with the automorphism group isomorphic to the non-split extension $3^{23} \cdot Co_2$, where the Co_2-module given by 3^{23} can be obtained as a submodule of codimension 1 in $\Lambda/3\Lambda$, Λ the Leech lattice.*

As it was already mentioned, $\mathcal{G}$ is 3-simply connected [8]. It implies that in a proper flag-transitive 2-cover of $\mathcal{G}$ every rank 3 residual P-geometry must be isomorphic to $\mathcal{G}(3 \cdot M_{22})$, and consequently every rank 4 residual P-geometry is isomorphic to $\mathcal{G}(3^{23} \cdot Co_2)$. Let $\tilde{\mathcal{G}}$ be the universal 2-cover of $\mathcal{G}$, and $\tilde{G}$ its automorphism group which is the full preimage in $\mathrm{Aut}\, \tilde{\mathcal{G}}$ of $G = BM$. In view of Proposition 3.6, $\tilde{\mathcal{G}}$ is a proper 2-cover of $\mathcal{G}$.

The geometry $\mathcal{G}$ contains a family of rank 4 (including types 1 to 4 of the above diagram) subgeometries, which are the classical symplectic spaces over $GF(2)$. The stabilizer in G of a subgeometry $\mathcal{S}$ is a subgroup $2^{9+16}.\mathrm{Sp}_8(2)$, which induces on $\mathcal{S}$ its full automorphism group $\mathrm{Sp}_8(2)$. The diagram of $\mathcal{S}$ is C_4:



For each element $x \in \mathcal{G}^4$, there is exactly one symplectic subgeometry $\mathcal{S}_x$ containing x. As it follows from the diagrams, $\mathcal{S}_x$ contains all elements of types 1, 2, 3 incident to x. The general information about subgeometries in P-geometries can be found in Section 3 of [11].

Let F be a flag of $\mathcal{S}$ of type $\{1,2\}$. Then $\mathrm{res}_{\mathcal{G}}(F) \cong \mathcal{G}(M_{22})$, while $\mathrm{res}_{\mathcal{S}}(F)$ is isomorphic to the generalized quadrangle of order $(2,2)$, having $S_6 \cong \mathrm{Sp}_4(2)$ as its automorphism group. Let $\tilde{F}$ be a flag in $\tilde{\mathcal{G}}$, which maps onto F, and $\tilde{\mathcal{S}}$ the connected component of the preimage of $\mathcal{S}$, containing $\tilde{F}$. As it was said above, $\mathrm{res}_{\tilde{\mathcal{G}}}(\tilde{F}) \cong \mathcal{G}(3 \cdot M_{22})$. It means that $\mathrm{res}_{\tilde{\mathcal{S}}}(\tilde{F})$ is the triple cover (actually, 1-cover) of the generalized qudrangle for S_6. This edge is denoted by  so that the diagram of $\tilde{S}$ is $\tilde{C}_4$:

All such 1-covers of symplectic geometries over $GF(2)$ (for any rank) were determined in [10], where they were called *the symplectic type T-geometries*. For each value of rank there is exactly one such geometry. We formulate the result from [10] in case of rank 4.

LEMMA 4.2: *Let S be the rank 4 symplectic geometry over $GF(2)$ and $\tilde{S}$ its flag-transitive 1-cover, having the diagram $\tilde{C}_4$. Then $\tilde{S}$ is uniquely determined and its only flag-transitive automorphism group is isomorphic to the non-split extension $3^{35} \cdot \mathrm{Sp}_8(2)$.*

Clearly, the subgeometries $\tilde{S}$ inherit the property of the symplectic subgeometries of $\mathcal{G}$ that every element of type 4 belongs to exactly one subgeometry. We state this as

LEMMA 4.3: *The universal 2-cover $\tilde{\mathcal{G}}$ of $\mathcal{G} = \mathcal{G}(BM)$ contains a family $\mathcal{ST}$ of rank 4 symplectic type T-subgeometries, such that every element $x \in \tilde{\mathcal{G}}^4$ belongs to exactly one subgeometry $\mathcal{S}_x \in \mathcal{ST}$. All elements of types 1, 2, 3 incident to x belong to $\mathcal{S}_x$. The stabilizer of $\mathcal{S} \in \mathcal{ST}$ induces on $\mathcal{S}$ the group $3^{35} \cdot \mathrm{Sp}_8(2)$.*

A further property of the subgeometries from $\mathcal{ST}$ will be stated in the next section (cf. Lemma 5.1).

5. Bounding the kernel

In this section, in order to simplify the notation, we denote by $\mathcal{G}$ the universal 2-cover of $\mathcal{G}(BM)$, rather than $\mathcal{G}(BM)$ itself. The only exception is made for the main statement of the section, in which the notation agrees with that of the rest of the paper. Similarly, in this section G denotes not BM, but its full preimage in $\mathrm{Aut}\,\mathcal{G}$.

Let K be the kernel of the natural homomorphism of G onto BM, i.e., it is the group of deck transformations of $\mathcal{G}$ with respect to the natural mapping $\mathcal{G} \to \mathcal{G}(BM)$. By Lemma 4.1(1) the stabilizer H_2 of an element $x \in \mathcal{G}^2$ has the structure $[2^{32}].(S_3 \times 3 \cdot \mathrm{Aut}\, M_{22})$. It means that the intersection $K_x = K \cap H_2$ is of order 3. Since $\mathcal{G}$ is 3-simply connected, K is generated by the subgroups K_x, for all $x \in \mathcal{G}^2$.

By Lemma 4.3, if x ranges within a subgeometry $\mathcal{S} \in \mathcal{ST}$, the corresponding subgroups K_x generate an abelian group 3^{35}. Let $K_{\mathcal{S}}$ denote this subgroup. Then the group (or, rather, $\mathrm{Sp}_8(2)$-module) $K_{\mathcal{S}}$ has the following property, proved in [10].

LEMMA 5.1: *Let $\mathcal{S} \in \mathcal{ST}$ and $x \in \mathcal{S}^4$. Then the* 35 *subgroups K_y, $y \in \mathrm{res}(x)^2$, span $K_{\mathcal{S}}$.*

For $y \in \mathcal{G}$ of type $\neq 2$, let K_y be defined as $\langle K_x \colon x \in \mathrm{res}(y)^2 \rangle$.

LEMMA 5.2: (1) *For $y \in \mathcal{G}^1$, the subgroup K_y is abelian of rank* 23.

(2) *Let $y \in \mathcal{G}$ be of type* $3, 4$, *or* 5. *Then K_y is abelian. The subgroups K_x, $x \in \mathrm{res}(y)^2$, are linearly independent in K_y. In particular, the rank of K_y is equal to* $7, 35$ *or* 155, *if the type of y is* $3, 4$ *or* 5, *respectively.*

Proof: The statement (1) follows from Lemma 4.1(2). For y of type 3 and 4, the statement (2) follows from Lemma 5.1. Suppose y is of type 5. Consider two elements x_1, x_2 of type 2, incident to y. Since $\mathrm{res}(y)$ is a projective space, there is an element u of type 4, incident to y, x_1 and x_2. By the above, K_{x_1} and K_{x_2} commute. To prove linear independence, consider the action of $Q = O_2(G_y)$ on K_y. Since Q is the kernel of the action of G_y on $\mathrm{res}(y)$, Q stabilizes each $K_x, x \in \mathrm{res}(y)^2$. Let $z \in \mathrm{res}(y)^4$ and $\mathcal{S} = \mathcal{S}_z$. By Lemma 5.1, Q stabilizes $K_{\mathcal{S}}$. Comparing the orders of Q (i.e., 2^{30}) and $O_2(G_{\mathcal{S}})$ (i.e., 2^{9+16}) we see that Q induces on K_y a group of order at least $2^5 > 2$. Since G_y acts on $\mathrm{res}(y)^2$ primitively, the subgroups $K_x, x \in \mathrm{res}(y)^2$, are linearly independent. ∎

LEMMA 5.3: *Let $x \in \mathcal{G}^1$, $y \in \mathrm{res}(x)^i$ for $i = 3, 4, 5$. Then* $\mathrm{rank}(K_x \cap K_y) = 3, 7, 15$, *respectively.*

Proof: The lower bounds are evident, since these numbers are just the numbers of elements in $\mathrm{res}(x,y)^2$ in the respective cases. To obtain the upper bounds consider the action of $H = G_x \cap G_y$ on K_y. Since H contains $O_2(G_y)$, an argument as in the preceding lemma shows that the irreducible components of K_y with respect to action of H arise from the orbits of H on $\mathrm{res}(y)^2$. Consequently, we obtain the following decompositions of K_y's: 3+4, 7+28 and 15+140, respectively. For $i = 4, 5$, the statement of lemma now follows from Lemma 5.2(1). Let $i = 3$ and assume $K_y \le K_x$. Choose $y_1 \in \mathrm{res}(y)^5$. Since $\mathrm{res}(y)^2 \subset \mathrm{res}(y_1)^2$, we conclude that $K_x \cap K_{y_1}$ has dimension larger than 7 or 15 as previously established, a contradiction. ∎

LEMMA 5.4: *Let $z \in \tilde{\mathcal{G}}^4$ and $\{y_1, y_2\} = \mathrm{res}(z)^5$. Then $K_{y_1} \cap K_{y_2} = K_z$.*

Proof: By definition, $K_z \subseteq K_{y_1} \cap K_{y_2}$. Let $y = y_1$ or y_2. Consider the action of $H = G_z \cap G_y$ on K_y. As above, we can obtain the decomposition 35+120 of K_y. Hence $K_{y_1} \cap K_{y_2} = K_z$, unless $K_{y_1} = K_{y_2}$, which is a contradiction. ∎

LEMMA 5.5: *In the situation of Lemma 5.4, let $x \in \mathrm{res}(z)^1$. Then $K_x \le \langle K_{y_1}, K_{y_2} \rangle$.*

Proof: Let $U = K_x \cap \langle K_{y_1}, K_{y_2} \rangle$. Then $\mathrm{rank}(U) \ge \mathrm{rank}(K_{y_1} \cap K_x) + \mathrm{rank}(K_{y_2} \cap K_x) - \mathrm{rank}(K_{y_1} \cap K_{y_2} \cap K_x) = 15 + 15 - 7 = 23 = \mathrm{rank}(K_x)$. ∎

LEMMA 5.6: *Let $v \in \mathcal{G}^2$ and $z_1, z_2 \in \mathrm{res}(v)^1$. Then K_{z_1} and K_{z_2} commute.*

Proof: Let $z = z_1$ or z_2. Since G_z induces on K_z the group Co_2, $G_z \cap G_v$ induces on K_z the group $2^{10} : \mathrm{Aut}\, M_{22}$. Finally, the group $G_{z_1} \cap G_{z_2}$, which has index 2 in $G_z \cap G_v$, induces on K_z a group, having section $2^{10} : M_{22}$. Now, it is clear that K_z, with respect to the action of $K_{z_1} \cap K_{z_2}$, has irreducible components of dimensions 1 and 22. The component of dimension 1 is K_v, which is common for K_{z_1} and K_{z_2}. Therefore, in order to prove the lemma, we need only to find an element in $K_{z_1} \setminus K_v$, commuting with K_{z_2}.

Consider an element $u \in \mathrm{res}(v)^4$ and let $\{y_1, y_2\} = \mathrm{res}(u)^5$. By Lemmas 5.2(2) and 5.4, K_u is in the center of $\langle K_{y_1}, K_{y_2} \rangle$. By Lemma 5.5, the latter group contains K_{z_2}, so that $K_{z_1} \cap K_u$ commutes with K_{z_2}. By Lemma 5.3, $\mathrm{rank}(K_{z_1} \cap K_u) = 7$, so the result follows. ∎

LEMMA 5.7: *Let $y \in \mathcal{G}^5$, $U_1(y) = \langle K_x | x \in \mathrm{res}(y)^1 \rangle$ and $U_2(y) = \langle K_y, K_z | z \in \mathcal{G}^5, \{y, z\} = \mathrm{res}(v)^5$ for some $v \in \mathcal{G}^4 \rangle$. Then $U_1(y) = U_2(y)$.*

Remark: By Lemma 5.6 and the axioms of projective space, $U_1(y)$ is abelian.

Proof: (i) $U_1(y) \le U_2(y)$: Suppose $x \in \mathrm{res}(y)^1$. Take $v \in \mathrm{res}(y, x)^4$ and let $\{y, z\} = \mathrm{res}(v)^5$. By Lemma 5.5, we have $K_x \le \langle K_y, K_z \rangle \le U_2(y)$.

(ii) $U_2(y) \leq U_1(y)$: Let $v \in \mathrm{res}(y)^4$ and $\{y,z\} = \mathrm{res}(v)^5$. Let $a \in \mathrm{res}(z)^2$. Then, in the residue of z, v is a hyperplane, while a is a line. Hence a and v have a point $x \in \mathrm{res}(v)^1$ in common. Clearly, x is incident to y. We have $K_a \leq K_x \leq U_1(y)$. Since by definition K_z is generated by all such K_a, we have $U_2(y) \leq U_1(y)$. ∎

Let us consider the graph Γ, defined on the set $\mathcal{G}^5$, where two elements are adjacent if and only if they are incident to a common element of type 4, i.e., $V(\Gamma) = \mathcal{G}^5$ and $E(\Gamma) = \mathcal{G}^4$. Let $\Gamma_i(y)$ denote the set of vertices at distance i from a vertex y. Put $V_i(y) = \langle K_z | z \in V(\Gamma), d(z,y) \leq i\rangle$. Then $V_0(y) = K_y$ and $V_1(y) = U_1(y) = U_2(y)$ by Lemma 5.7.

LEMMA 5.8: $\mathrm{rank}(V_1(y)/V_0(y)) \leq 248$.

Proof: By Lemma 5.7, $V_1(y)$ is generated by 31 subgroups $K_z, z \in \mathrm{res}(y)^1$, each having dimension 23 and intersecting $V_0(y)$ by a subspace of dimension 15 (Lemma 5.3). Hence $\mathrm{rank}(V_1(y)/V_0(y)) \leq 31 \cdot 8 = 248$. ∎

LEMMA 5.9: *Let $z \in \Gamma_1(y)$. Then* $\mathrm{rank}(V_1(z)/\langle V_0(y), V_0(z)\rangle) \leq 128$.

Proof: Let $v \in \mathcal{G}^4$ be incident to both y and z. The group $V_1(z)$ is generated by the subgroups K_x for $x \in \mathrm{res}(z)^1$. If x is incident to v, then $K_x \leq \langle V_0(y), V_0(z)\rangle$ by Lemma 5.5. There are exactly 16 elements x, which are not adjacent to v, each of them giving contribution of rank at most 8 (indeed, by Lemma 5.2 (1) the rank of K_x equals 23, while the rank of $K_x \cap K_z$ is 15 by Lemma 5.3). ∎

COROLLARY 5.10: *If $V_2(y)$ is abelian, then* $\mathrm{rank}(V_2(y)/V_1(y)) \leq 31 \cdot 128 = 3968$.

LEMMA 5.11: *Let $y \in \mathcal{G}^5$, $u \in \mathrm{res}(y)^3$, $\{v_1, v_2, v_3\} = \mathrm{res}(y,u)^4$ and, for $i = 1,2,3$, let $\{y, z_i\} = \mathrm{res}(v_i)^5$. Then $V_1(y) = \langle V_0(y), V_0(z_1), V_0(z_2), V_0(z_3)\rangle$.*

Proof: In the residue of y the elements v_1, v_2 and v_3 are three hyperplanes having a subspace of codimension 2 in common. Therefore, every point $x \in \mathrm{res}(y)^1$ is incident to z_i for some i. By Lemma 5.5, $K_x \leq \langle V_0(y), V_0(z_i)\rangle$. ∎

LEMMA 5.12: $V_3(y) = V_2(y)$.

Proof: Let $z \in \Gamma_2(y)$. Then we can find an element $u \in \mathcal{G}^3$, which is incident to both y and z. Let $\{v_1, v_2, v_3\} = \mathrm{res}(z,u)^4$ and $\{z, z_i\} = \mathrm{res}(v_i)^5$ for $i = 1,2,3$.

Since $\mathrm{res}(u)^{4,5}$ is the dual Petersen geometry, and since the Petersen graph has diameter 2, one has $V_1(z) \leq \langle V_0(z), V_0(z_1), V_0(z_2), V_0(z_3)\rangle \leq V_2(y)$. The first containment is forced by Lemma 5.11. ∎

COROLLARY 5.13: *For every $y \in \mathcal{G}^5$ one has $K = V_2(y)$.*

Proof: Follows from Lemma 5.12 and connectedness of Γ. ∎

LEMMA 5.14: *K is abelian.*

Proof: By Corollary 5.13, $K = V_2(y)$. Let us show that $V_0(y)$ is in the center of K. Let $z \in \mathcal{G}^5$, $d(y,z) \leq 2$ and $a \in \mathcal{G}^5$ be such that $d(a,y) \leq 1$ and $d(a,z) \leq 1$. By the remark after Lemma 5.7, $V_1(a)$ is abelian. Hence $V_0(y)$ and $V_0(z)$ commute.

We have proven that $V_0(y)$ is in the center of K. Since $K = \langle V_0(y) : y \in \mathcal{G}^5\rangle$, K is abelian. ∎

Now the main result of the section. Recall that here we return to the general notation of the paper.

PROCLAIM 5.15: *Let $\mathcal{G} = \mathcal{G}(BM)$, $\tilde{\mathcal{G}}$ be its universal 2-cover and K be the group of deck transformation of $\tilde{\mathcal{G}}$ with respect to the mapping $\tilde{\mathcal{G}} \to \mathcal{G}$. Then K is elementary abelian 3-group of rank at most 4371.*

Proof: By Lemma 5.14, K is abelian. By Lemmas 5.2(2), 5.8 and Corollary 5.10, $\mathrm{rank}(K) \leq 155 + 248 + 3968 = 4371$. ∎

6. The end of the proof

In the last section we complete the proof of Main Theorem. Let $\mathcal{G} = \mathcal{G}(BM)$, $\tilde{\mathcal{G}}$ be its universal 2-cover, and $\tilde{G}$ be the full preimage in $\mathrm{Aut}\,\tilde{\mathcal{G}}$ of the group $G = BM$. Let K be the group of deck transformations of $\tilde{\mathcal{G}}$ with respect to the mapping $\tilde{\mathcal{G}} \to \mathcal{G}$. Then K is the kernel of the natural homomorphism of $\tilde{G}$ onto G.

So far we proved that $K \neq 1$ (Proposition 3.6) and that it is elementary abelian 3-group of rank at most 4371 (Proposition 5.15). Since $\tilde{G}$ acts on $\tilde{\mathcal{G}}$ flag-transitively, it remains to prove that:

(a) The G-module K is irreducible of dimension exactly 4371.

(b) The extension $\tilde{G} = K.G$ is non-split.

Let us start with (a).

Consider the action on K of the subgroup $H \cong 2^{1+22} \cdot Co_2 \leq G$. Let $Q = O_2(H)$ and z be the central involution of H. Clearly, G acts on K non-trivially, and consequently H acts on K faithfully. It is well-known that, over any field of odd characteristic, an extraspecial group 2^{1+2s} has exactly one irreducible faithful module, and that that module has dimension 2^s. In our case it means that $\dim [K, z] \geq 2048$. In Section 3, proof of Lemma 3.12 (where H, Q appeared under the names H_1, Q_1) we saw that Q contains a non-central element z' which is G-conjugate to z. If $C_K(z) = C_K(Q)$ then $C_K(z') > C_K(z)$, a contradiction since z and z' are G-conjugate. Therefore, K contains an irreducible H-submodule K_0, such that $[z, K_0] = 1$ and $[Q, K_0] \neq 1$. Since Q is normal in H and induces on K_0 the elementary abelian 2-group $Q/\langle z \rangle$, Clifford's theorem implies that $\dim K_0$ is at least the minimal length of orbit of $H/Q \cong Co_2$ on the set of hyperplanes of $Q/\langle z \rangle$. Since the Co_2-module $Q/\langle z \rangle$ is self-dual, Table 1 from [16] shows that $\dim K_0 \geq 2300$. Finally, H is the stabilizer of a certain element $x \in \mathcal{G}^1$. In particular, H stabilizes the 23-dimensional subspace $K_{\tilde{x}}$, where $\tilde{x}$ belongs to the preimage of x in $\tilde{\mathcal{G}}$ and $K_{\tilde{x}}$ is defined as in Section 5. Clearly, $K_{\tilde{x}} \leq C_K(Q)$. Now, summing up the dimensions of $[K, z]$, K_0 and $K_{\tilde{x}}$, we obtain that $\dim K \geq 2048 + 2300 + 23 = 4371$. Since also $\dim K \leq 4371$, we finally establish $\dim K = 4371$.

Continuing the above, suppose K is not irreducible with respect to the action of G. From the decomposition $K = [K, z] \oplus K_0 \oplus K_{\tilde{x}}$ we see that, as a G-module, K has no trivial composition factors. If U is a faithful G-module then, substituting K by U in the previous paragraph, we obtain at least that $\dim U \geq 2048 + 2300$. So a reducible G-module without trivial composition factors must has dimension at least twice of the given sum. Therefore, K is irreducible.

Now let us turn to (b). Let H, Q, x and $\tilde{x}$ be as above. Let $\tilde{H}$ be the stabilizer of $\tilde{x}$ in $\tilde{G}$, and $\tilde{Q} = O_2(\tilde{H})$. Clearly, the natural homomorphism $\tilde{G} \to G$ maps $\tilde{H}$ and $\tilde{Q}$ onto H and Q, respectively. It follows from Lemma 4.1(2) that the extension $\tilde{H} = K_{\tilde{x}}.H$ is non-split. On the other hand, we proved above that $C_K(Q) = K_{\tilde{x}}$ and hence $\tilde{H} = N_{\tilde{G}}(\tilde{Q})$. The latter group splits over $N_{\tilde{G}}(\tilde{Q}) \cap K = C_K(Q) = K_{\tilde{x}}$ whenever $\tilde{G}$ splits over K. This gives (b), and the proof of Main Theorem is complete. ∎

One might observe that 4371, i.e. the dimension of K, coincides with the minimal degree of a non-trivial irreducible representation of $G = BM$ over $\mathbf{C}$. So the following question seems to be quite natural: Is K isomorphic to the module obtained by taking modulo 3 the rational-valued representation of G of degree 4371? We conclude this paper with rather informal comments demonstrating that this question must be answered affirmatively. Namely, we claim that G has exactly one irreducible module of degree 4371 over $GF(3)$.

Let U be such a module. As above we can obtain a decomposition $U = U_1(x) \oplus U_2(x) \oplus U_3(x)$ with respect to the action of $H = G_x$. Here $U_i(x)$ has dimension 2048, 2300, 23 for $i = 1, 2, 3$, respectively. Let $y \in \mathrm{res}(x)^2$, let N be the stabilizer of y in G, and $R = O_2(N)$. Then $R \cap Q$ covers a hyperplane in $Q/\langle z \rangle$, belonging to the H/Q-orbit of length 46575. In particular, R fixes no non-trivial vector in $U_1(x)$ and $U_2(x)$. Consider now the action of R on $U_3(x)$. The image of $N \cap H$ in $H/Q \cong Co_2$ is a maximal subgroup $2^{10} : \mathrm{Aut}\, M_{22}$. It is easy to show that $U_3(x)$ splits as 1+22 under the action of $N \cap H$, and to determine the character of $U_3(x)$ on the involutions of Co_2. Finally it brings us to the following conclusion: for every $y \in \mathcal{G}^2$ the subspace $U_y = C_U(O_2(G_y))$ has dimension 1; an element of G_y inverts U_y if and only if it induces an outer automorphism on the section M_{22} of G_y.

Take now a symplectic subgeometry $\mathcal{S}$ passing through x and y. Let D be the stabilizer of $\mathcal{S}$ in G. The image of $D \cap H$ in H/Q is a maximal subgroup $2^{1+8}.\mathrm{Sp}_6(2)$. Considering its action on $U_3(x)$ we establish the decomposition 7+16 for $U_3(x)$. As a consequence we obtain that the 63 subspaces $U_y, y \in \mathrm{res}_{\mathcal{S}}(x)^2$, span the 7-dimensional component, which is simply $C_{U_3(x)}(O_2(D))$. Now, according to [12], the group $\mathrm{Sp}_6(2)$ has exactly 1 non-trivial 7-dimensional module in characteristic 3, namely, the E_7-lattice taken modulo 3. By inspecting the latter module we establish that $\mathrm{Sp}_6(2)$, acting on the set of 1-spaces of this module, has exactly one orbit of length 63. Consequently, the 7-dimensional module has a unique presentation in terms of its subspaces $U_y, y \in \mathrm{res}_{\mathcal{S}}(x)^2$. It follows that the module U is a factormodule of a unique maximal (or rather, universal) G-module $\hat{U}$ over $GF(3)$ such that

(1) $\hat{U}$ is spanned by 1-spaces $\hat{U}_y$, where $y \in \mathcal{G}^2$, $\hat{U}_y$ is invariant under the action of the stabilizer N of y in G, and an element of N inverts $\hat{U}_y$ whenever it induces an outer automorphism on the section M_{22} of N;

(2) the subspace $\langle \hat{U}_y | y \in \mathrm{res}_{\mathcal{S}}(x)^2 \rangle$ has dimension 7 for every $x \in \mathcal{G}^1$ and every symplectic subgeometry $\mathcal{S}$ on x.

Now, the property (2) can be used to prove that the subspace $\hat{U}_x = \langle \hat{U}_y | y \in \mathrm{res}_{\mathcal{G}}(x)^2 \rangle$ has dimension 23, while the subspace $\hat{U}_S = \langle \hat{U}_y | y \in \mathcal{S}^2 \rangle$ has dimension 35 and, moreover, has the property indicated in Lemma 5.1. Essentially, the proof of the first statement was given in [15], while the second statement was proved in [10]. Since the arguments in Section 5 were based only on Lemma 5.1 and Lemma 5.2(1), those arguments can be applied to $\hat{U}$ to establish finally that $\dim \hat{U} \leq 4371$, and, consequently, $U \cong \hat{U} \cong K$.

ACKNOWLEDGEMENT: We thank the referee of this paper for his many helpful comments.

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